FACTORIZATION PROBLEM ON THE HILBERT-SCHMIDT GROUP AND THE CAMASSA-HOLM EQUATION

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Abstract. In this paper, we solve the Camassa-Holm equation for a relatively large class of initial data by using a factorization problem on the Hilbert-Schmidt group.

1. Introduction.

The Camassa-Holm (CH) equation is a model of long waves in shallow water. Since its introduction in 1993 by Camassa and Holm [CH], the equation has received considerable attention and its various aspects were studied using a variety of methods. (See, for example, [BF],[BSS], [Con], [CM], [CS], [GH], [M], [XZ] and the references therein.)

In contrast to the KdV equation, the CH equation admits breaking solutions. However, a relatively large class of initial data which give rise to global solutions has also been identified in [C1] and independently in [Con]. In the paper [C1] and its more elaborate version [C2], the initial value problem for the CH equation was analyzed through its characteristic formulation. It is in this context that an isospectral problem in the form of an integro-differential equation was discovered for the Lagrangian version of the integrable PDE. This remarkable fact has led, in particular, to a particle method for numerically solving the CH equation.(See [CHL] for subsequent development of this particle method.)

The main goal of this work is to show that the integro-differential equation in the afore-mentioned papers of Camassa is in fact exactly solvable, in the sense that a formula for its solution can be written down. The upshot of this, as the reader will see, is that we can integrate both the Lagrangian version and the Eulerian version of the CH equation explicitly. The solution of the integro- differential equation is based on a factorization problem on the Hilbert-Schmidt group which we will

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introduce. On the other hand, we will show that the factorization problem on the Hilbert-Schmidt group can be reduced to solving a family of Fredholm integral equations, and this can be achieved by using regularized Fredholm determinants and Fredholm first minors. The reader will see that the Lax equation which corresponds to the integro-differential equation is in some sense an infinite dimensional analog of the Toda flow on $n \times n$ matrices (cf. [DLT]). That such an analog is connected with the Camassa-Holm equation is rather remarkable and is responsible for the elementary method of solving the equation here.

The paper is organized as follows. In Section 2, we begin by introducing a class of integrable isospectral deformations of Hilbert-Schmidt operators on $L^2(\mathbb{R})$ using the r-matrix approach, then we discuss the underlying Lie groups and coadjoint orbits. Since the Hilbert-Schmidt operators on $L^2(\mathbb{R})$ are given by integral operators with kernels in $L^2(\mathbb{R}^2)$, this leads naturally to a class of integrable integro-differential equations. In particular, for a special choice of Hamiltonian, the corresponding Lax equation gives rise to the integro-differential equation obtained in [C1], [C2]. In Section 3, we discuss the solution of the factorization problem on the Hilbert-Schmidt group and its application towards the integration of the integro-differential equation. Finally in Section 4, we show how to apply our results to obtain the explicit integration of the Lagrangian version and the Eulerian version of the Camassa-Holm equation.

To close this introduction, we remark that the Lagrangian version of the CH equation was also explicitly integrated in [M]. We note, however, that the assumption on the initial data and the method used in [M] are quite different from the one employed here.(See Remark 4.3 for the relationship between the spectral problems.)

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2. Classical r-matrices and integrable integro-differential equations.

In this section, we introduce a class of integrable integro-differential equations associated with isospectral deformations of Hilbert-Schmidt operators.

Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R})$ consisting of real-valued measurable functions on $\mathbb{R}$ that satisfy $\int_{\mathbb{R}} |f(x)|^2 \, dx < \infty$ with inner product

$$(f, g) = \int_{\mathbb{R}} f(x) g(x) dx \tag{2.1}$$

and let $\mathfrak{g}$ be the space of Hilbert-Schmidt operators on $\mathcal{H}$. It is well-known that [RS] $\boldsymbol{K} \in \mathfrak{g}$ if and only if there is a function $K \in L^2(\mathbb{R}^2)$ uniquely determined by $\boldsymbol{K}$ such that

$$(\boldsymbol{K}\varphi)(x) = \int_{\mathbb{R}} K(x, y)\varphi(y) \, dy. \tag{2.2}$$

In other words, the Hilbert-Schmidt operators on $\mathcal{H}$ are precisely the integral operators on $\mathcal{H}$ with L^2-kernels. Moreover, for $\boldsymbol{K} \in \mathfrak{g}$, the Hilbert-Schmidt norm is given by

$$\|\boldsymbol{K}\|_2^2 = \int_{\mathbb{R}^2} |K(x, y)|^2 \, dxdy. \tag{2.3}$$

Let $B(\mathcal{H})$ be the space of bounded operators on $\mathcal{H}$. We recall that if $T \in B(\mathcal{H})$, its adjoint T^* is defined by means of the relation

$$(T^*\varphi, \psi) = (\varphi, T\psi) \tag{2.4}$$

for all $\varphi \in \mathcal{H}$ and $\psi \in \mathcal{H}$. Specializing to the case where $\boldsymbol{A} \in \mathfrak{g} \subset B(\mathcal{H})$ with kernel $A(x, y)$, it is easy to show from (2.4) and Fubini's theorem that its adjoint $\boldsymbol{A}^*$ is the integral operator on $\mathcal{H}$ with kernel $A^*(x, y) = A(y, x)$. Thus $\boldsymbol{A}^*$ is also in $\mathfrak{g}$. In what follows, if $\boldsymbol{A} \in \mathfrak{g}$, we shall use the notation

$$\boldsymbol{A} \doteq A(x, y) \tag{2.5}$$

to mean that the integral operator $\boldsymbol{A}$ has kernel $A(x, y)$.

Proposition 2.1. *$\mathfrak{g}$ is a Hilbert Lie algebra with Lie bracket $[\cdot, \cdot]$ defined by*

$$\begin{aligned} [\boldsymbol{A}, \boldsymbol{B}] &= \boldsymbol{A} \circ \boldsymbol{B} - \boldsymbol{B} \circ \boldsymbol{A} \\ &\doteq \int_{\mathbb{R}} (A(x, z)B(z, y) - B(x, z)A(z, y)) \, dz \end{aligned} \tag{2.6}$$

and the inner product on $\mathfrak{g}$ *is the usual Hilbert-Schmidt inner product* $(\cdot,\cdot)_2$, *i.e.,*

$$\begin{aligned}(\boldsymbol{A},\boldsymbol{B})_2 &= tr(\boldsymbol{A}^*\circ \boldsymbol{B})\\ &= \int_{\mathbb{R}^2} A(x,y)B(x,y)\,dxdy.\end{aligned} \tag{2.7}$$

Moreover, $\mathfrak{g}$ *is equipped with the non-degenerate ad-invariant pairing*

$$(\boldsymbol{A},\boldsymbol{B}) = \int_{\mathbb{R}^2} A(x,y)B(y,x)\,dxdy. \tag{2.8}$$

Proof. Since the Hilbert-Schmidt operators is closed under the operations of addition, subtraction, and composition, it follows that the bracket operation in (2.6) is well-defined and it is clear that $[\cdot,\cdot]$ is a Lie bracket. On the other hand, it is well-known that $\mathfrak{g}$ with the inner product in (2.7) is a Hilbert space. Hence $\mathfrak{g}$ is a Hilbert Lie algebra. We shall leave the rest of the assertion to the reader as an exercise. $\square$

In addition to what we have above, we remark that as a special case of a general theorem (see, for example, [RS]), $\mathfrak{g}$ is a 2-sided ideal in $B(\mathcal{H})$. Now, let $\boldsymbol{I}\in B(\mathcal{H})$ be the identity operator, and let $GL(\mathcal{H})$ denote the group of invertible operators in $B(\mathcal{H})$. We define

$$G = GL(\mathcal{H})\bigcap(\boldsymbol{I}+\mathfrak{g}). \tag{2.9}$$

If $\boldsymbol{I}+\boldsymbol{K}\in G$, it is well-known that $(\boldsymbol{I}+\boldsymbol{K})^{-1}$ is also in G. (See, for example, [Sm].) Hence G is a group under the composition of operators. As a matter of fact, G is a Hilbert Lie group which integrates the Lie algebra $\mathfrak{g}$, the Hilbert manifold structure is being determined by the map $G\longrightarrow\mathfrak{g}:\boldsymbol{I}+\boldsymbol{K}\mapsto\boldsymbol{K}$ which is a bijection onto an open subset of $\mathfrak{g}$ consisting of Hilbert-Schmidt operators for which -1 is not an eigenvalue. We will call G the *Hilbert-Schmidt group.* In this case, the adjoint action of the group G on $\mathfrak{g}$ is given by the formula

$$Ad_G(g)\boldsymbol{K} = g\circ\boldsymbol{K}\circ g^{-1}. \tag{2.10}$$

Because the pairing on $\mathfrak{g}$ is ad-invariant, we also have

$$(g\circ\boldsymbol{A}\circ g^{-1},\boldsymbol{B}) = (\boldsymbol{A}, g^{-1}\circ\boldsymbol{B}\circ g). \tag{2.11}$$

On the other hand, the exponential map $\exp:\mathfrak{g}\longrightarrow G$ is given by the expression

$$\exp(\boldsymbol{K}) = \sum_{j=0}^{\infty}\frac{\boldsymbol{K}^j}{j!} \tag{2.12}$$

where the powers of $\boldsymbol{K}$ are defined recursively by

$$\boldsymbol{K}^0 = \boldsymbol{I}, \quad \boldsymbol{K}^j = \boldsymbol{K} \circ \boldsymbol{K}^{j-1}, \ j = 1, 2, \cdots . \tag{2.13}$$

The Hilbert Lie algebra $\mathfrak{g}$ has two distinguished Lie subalgebras $\mathfrak{l}$ and $\mathfrak{k}$, where $\mathfrak{l}$ consists of Volterra integral operators $\boldsymbol{A}$ of the form

$$(\boldsymbol{A}\varphi)(x) = \int_{-\infty}^{x} A(x,y)\varphi(y)\,dy \tag{2.14}$$

and $\mathfrak{k}$ consists of integral operators $\boldsymbol{B}$ for which

$$\boldsymbol{B}^* = -\boldsymbol{B}. \tag{2.15}$$

We will call $\mathfrak{l}$ the *lower triangular subalgebra* of $\mathfrak{g}$ and $\mathfrak{k}$ the *skew-symmetric subalgebra.*

Proposition 2.2. *We have*

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{k}.$$

Proof. Given $\boldsymbol{K} \in \mathfrak{g}$ with kernel $K(x,y)$, it is clear that

$$\boldsymbol{K} = \Pi_{\mathfrak{k}}\boldsymbol{K} + \Pi_{\mathfrak{l}}\boldsymbol{K}$$

where

$$\Pi_{\mathfrak{k}}\boldsymbol{K} \doteq K(x,y)\chi_{(x,\infty)}(y) - K(y,x)\chi_{(-\infty,x)}(y)$$

and

$$\Pi_{\mathfrak{l}}\boldsymbol{K} \doteq K(x,y)\chi_{(-\infty,x)}(y) + K(y,x)\chi_{(-\infty,x)}(y).$$

In the above expressions, $\chi_{(x,\infty)}(y)$ and $\chi_{(-\infty,x)}(y)$ are respectively the characteristic functions of (x,∞) and $(-\infty,x)$. Now it is straightforward to check that $\Pi_{\mathfrak{l}}\boldsymbol{K} \in \mathfrak{l}$ and $\Pi_{\mathfrak{k}}\boldsymbol{K} \in \mathfrak{k}$. Therefore, $\mathfrak{g} = \mathfrak{l} + \mathfrak{k}$. To show that the sum is direct, suppose $K(\cdot,\cdot)$ is the kernel of an operator which belongs to both $\mathfrak{l}$ and $\mathfrak{k}$. Then $K(x,y) = 0$ away from the diagonal and hence the corresponding operator is zero. This completes the proof. $\square$

Let $\Pi_{\mathfrak{l}}$ and $\Pi_{\mathfrak{k}}$ be the projection operators onto $\mathfrak{l}$ and $\mathfrak{k}$ respectively associated with the decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{k}$. Then it follows from [STS] that

$$R = \Pi_{\mathfrak{l}} - \Pi_{\mathfrak{k}} \tag{2.16}$$

is a classical r-matrix on $\mathfrak{g}$ satisfying the modified Yang-Baxter equation (mYBE)

$$[R(\boldsymbol{A}), R(\boldsymbol{B})] - R([R(\boldsymbol{A}), \boldsymbol{B}] + [\boldsymbol{A}, R(\boldsymbol{B})]) = -[\boldsymbol{A}, \boldsymbol{B}] \tag{2.17}$$

for all $\boldsymbol{A}$, $\boldsymbol{B} \in \mathfrak{g}$. Consequently, the formula

$$[\boldsymbol{A}, \boldsymbol{B}]_R = \frac{1}{2}([R(\boldsymbol{A}), \boldsymbol{B}] + [\boldsymbol{A}, R(\boldsymbol{B})]), \quad \boldsymbol{A}, \boldsymbol{B} \in \mathfrak{g} \tag{2.18}$$

defines a second Lie bracket on $\mathfrak{g}$ and we shall denote the associated Lie algebra by $\mathfrak{g}_R$. In what follows, we shall compute the dual maps of all linear operators on $\mathfrak{g}$ with respect to the pairing $(\cdot, \cdot)$ in (2.8).

Proposition 2.3. *If $\boldsymbol{A} \in \mathfrak{g}$ and $\boldsymbol{A} \doteq A(x,y)$, then*

$$\Pi_{\mathfrak{k}}^* \boldsymbol{A} \doteq (A(x,y) - A(y,x))\chi_{(-\infty,x)}(y)$$
$$\Pi_{\mathfrak{l}}^* \boldsymbol{A} \doteq A(x,y)\chi_{(x,\infty)}(y) + A(y,x)\chi_{(-\infty,x)}(y).$$

The proof is a straightforward calculation and so we skip the details. We shall equip $\mathfrak{g}_R^* \simeq \mathfrak{g}$ with the Lie-Poisson structure

$$\{ F_1, F_2 \}_R(\boldsymbol{K}) = (\boldsymbol{K}, [dF_1(\boldsymbol{K}), dF_2(\boldsymbol{K})]_R) \tag{2.19}$$

where F_1, $F_2 \in C^\infty(\mathfrak{g}_R^*)$, and $dF_i(\boldsymbol{K}) \in \mathfrak{g}$ is defined by the formula $\frac{d}{dt}\big|_{t=0} F_i(\boldsymbol{K} + t\boldsymbol{K}') = (dF_i(\boldsymbol{K}), \boldsymbol{K}')$, $i = 1, 2$.

The following result is a consequence of standard classical r-matrix theory. (See [STS] and [RSTS] for the general theory.)

Proposition 2.4. *(a) The Hamiltonian equations of motion generated by $F \in C^\infty(\mathfrak{g}_R^*)$ is given by*

$$\dot{\boldsymbol{K}} = \frac{1}{2}[R(dF(\boldsymbol{K})), \boldsymbol{K}] - \frac{1}{2}R^*[\boldsymbol{K}, dF(\boldsymbol{K})]. \tag{2.20}$$

In particular, for the Hamiltonian $H_j(\boldsymbol{K}) = \frac{1}{2(j+1)} tr(\boldsymbol{K}^{j+1})$, $j = 1, 2, \ldots,$ the corresponding equation is the Lax equation

$$\dot{\boldsymbol{K}} = \frac{1}{2}[\,\Pi_{\mathfrak{l}}\, \boldsymbol{K}^j, \boldsymbol{K}\,]. \tag{2.21}$$

(b) The family of functions $H_j(\boldsymbol{K})$, $j = 1, 2, \ldots$ Poisson commute with respect to $\{\cdot, \cdot\}_R$.

Proof. The Hamiltonian equation of motion (2.20) is obtained from (2.19) by a direct calculation. On the other hand, by using (2.11), we find that $Ad^*_G(g^{-1})\boldsymbol{A} = g \circ \boldsymbol{A} \circ g^{-1}$, $g \in G$. Since $tr(\boldsymbol{A} \circ \boldsymbol{B}) = tr(\boldsymbol{B} \circ \boldsymbol{A})$ for any $\boldsymbol{B} \in B(\mathcal{H})$ and any trace class operator $\boldsymbol{A}$, it follows that $H_j(Ad^*_G(g^{-1})\boldsymbol{K}) = H_j(\boldsymbol{K})$, $g \in G$. By classical r-matrix theory, we then conclude that the family of functions $H_j(\boldsymbol{K})$, $j = 1, 2, \ldots$ Poisson commute with respect to $\{\cdot,\cdot\}_R$. The equation of motion for H_j now follows from (2.20) as the invariance property $H_j(Ad^*_G(g^{-1})\boldsymbol{K}) = H_j(\boldsymbol{K})$ implies $[\boldsymbol{K}, dH_j(\boldsymbol{K})] = 0$. This completes the proof. □

Let

$$\mathfrak{p} = \{\boldsymbol{K} \in \mathfrak{g} \mid \boldsymbol{K} = \boldsymbol{K}^*\}. \tag{2.22}$$

Corollary 2.5. *(a) $\mathfrak{p}$ is a Poisson submanifold of $(\mathfrak{g}^*_R, \{\cdot,\cdot\}_R)$. Hence eqn. (2.21) with $\boldsymbol{K} \in \mathfrak{p}$ is Hamiltonian with respect to the induced Poisson structure on $\mathfrak{p}$.*
(b) For the Hamiltonian $H_1(\boldsymbol{K}) = \frac{1}{4}(\boldsymbol{K}, \boldsymbol{K})$, the evolution of the kernel $K(x,y;t)$ corresponding to $\boldsymbol{K}$ is given by the integro-differential equation

$$\begin{aligned}\dot{K}(x,y\,;t) =& \frac{1}{2}\int_{-\infty}^{x} K(x,z\,;t)K(z,y\,;t)\,dz - \frac{1}{2}\int_{y}^{\infty} K(x,z\,;t)K(z,y\,;t)\,dz \\ &+ \frac{1}{2}\int_{-\infty}^{x} K(z,x\,;t)K(z,y\,;t)\,dz - \frac{1}{2}\int_{y}^{\infty} K(x,z\,;t)K(y,z\,;t)\,dz.\end{aligned} \tag{2.23}$$

In the special case when $\boldsymbol{K}$ belongs to the Poisson submanifold $\mathfrak{p}$, the corresponding kernel $K(x,y;t)$ is symmetric. In this case, the above equation reduces to

$$\begin{aligned}\dot{K}(x,y\,;t) &= \int_{-\infty}^{x} K(x,z\,;t)K(z,y\,;t)\,dz - \int_{y}^{\infty} K(x,z\,;t)K(z,y\,;t)\,dz \\ &= \int_{-\infty}^{y} K(x,z\,;t)K(z,y\,;t)\,dz - \int_{x}^{\infty} K(x,z\,;t)K(z,y\,;t)\,dz.\end{aligned} \tag{2.24}$$

Proof. (a) From (2.20), the Hamiltonian vector field generated by F can be rewritten in the form

$$X_F(\boldsymbol{K}) = [\boldsymbol{K}, \Pi_{\mathfrak{k}}(dF(\boldsymbol{K}))] - \Pi^*_{\mathfrak{l}}[\boldsymbol{K}, dF(\boldsymbol{K})].$$

From the expression for $\Pi^*_{\mathfrak{l}}$ in Proposition 2.3, it is clear that the second term in the above expression is always in $\mathfrak{p}$. On the other hand, if $\boldsymbol{K} \in \mathfrak{p}$, then it is easy to check that $[\boldsymbol{K}, \Pi_{\mathfrak{k}}(dF(\boldsymbol{K}))]$ is also in $\mathfrak{p}$. Consequently, we have $X_F(\boldsymbol{K}) \in \mathfrak{p}$ for

$\boldsymbol{K} \in \mathfrak{p}$ and this shows that $\mathfrak{p}$ is a Poisson submanifold of $(\mathfrak{g}_R^*, \{\cdot,\cdot\}_R)$.

(b) We have

$$(\Pi_{\mathfrak{l}}\boldsymbol{K} \circ \boldsymbol{K})\varphi(x) = \int_{\mathbb{R}} \left(\int_{-\infty}^{x} (K(x,z\,;t) + K(z,x\,;t))K(z,y\,;t)\,dz \right) \varphi(y)\,dy$$

while

$$(\boldsymbol{K} \circ \Pi_{\mathfrak{l}}\boldsymbol{K})\varphi(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(x,z\,;t)(K(z,y\,;t) + K(y,z\,;t))\chi_{(-\infty,z)}(y)\,dz \right) \varphi(y)\,dy.$$

Therefore, the evolution of the kernel $K(x,y\,;t)$ is given by

$$\begin{aligned} 2\dot{K}(x,y\,;t) &= \int_{-\infty}^{x} K(x,z\,;t)K(z,y\,;t)\,dz - \int_{y}^{\infty} K(x,z\,;t)K(z,y\,;t)\,dz \\ &\quad + \int_{-\infty}^{x} K(z,x\,;t)K(z,y\,;t)\,dz - \int_{y}^{\infty} K(x,z\,;t)K(y,z\,;t)\,dz. \end{aligned}$$

□

For our next remark and the discussion in Section 3, we introduce the Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ which consists of Volterra integral operators $\boldsymbol{B}$ of the form

$$(\boldsymbol{B}\varphi)(x) = \int_{x}^{\infty} B(x,y)\varphi(y)\,dy. \tag{2.25}$$

Remark 2.6. (a) From the definition of $\mathfrak{k}$ and $\mathfrak{l}$, and from equation (2.21), it is clear that what we are dealing with here is in some sense an infinite dimensional analog of the Toda flows on $n \times n$ matrices (cf. [DLT]).

(b) A different decomposition of the Hilbert Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u} \tag{2.26}$$

with associated projection maps $\Pi_- : \mathfrak{g} \longrightarrow \mathfrak{l}$ and $\Pi_+ : \mathfrak{g} \longrightarrow \mathfrak{u}$. (We will use these projection maps in Section 3.) Indeed, if we consider the r-matrix $R = \Pi_- - \Pi_+$ associated with this splitting and equip the corresponding $\mathfrak{g}_R^*$ with the Lie-Poisson structure, then the evolution of the *general kernel* $K(x,y;t)$ under the Hamiltonian flow generated by $\frac{1}{2}(\boldsymbol{K},\boldsymbol{K})$ is given by the equation

$$\begin{aligned} \dot{K}(x,y\,;t) &= \int_{-\infty}^{x} K(x,z\,;t)K(z,y\,;t)\,dz - \int_{y}^{\infty} K(x,z\,;t)K(z,y\,;t)\,dz \\ &= \int_{-\infty}^{y} K(x,z\,;t)K(z,y\,;t)\,dz - \int_{x}^{\infty} K(x,z\,;t)K(z,y\,;t)\,dz. \end{aligned}$$

Clearly, eqn.(2.24) is a special case of this. Note, however, that $\mathfrak{p}$ is not a Poisson submanifold any more with this choice of r-matrix and the corresponding Lie-Poisson structure. Indeed, the Hamiltonian vector field generated by a *general function* F is now of the form $[\boldsymbol{K}, \Pi_+(dF(\boldsymbol{K}))] - \Pi_-^*[\boldsymbol{K}, dF(\boldsymbol{K})]$. Clearly, this is not necessarily in $\mathfrak{p}$ for $\boldsymbol{K} \in \mathfrak{p}$. Thus from the Hamiltonian point of view, the r-matrix in (2.16) is the correct choice. For the relation between (2.24) and the Camassa-Holm equation, and the coadjoint orbit picture, we refer the reader to the discussion in Section 4 preceding Remark 4.1, Remark 4.2 and Proposition 2.8 below for details.

In the rest of the section, we shall describe the symplectic leaves of the Lie-Poisson structure $\{\cdot,\cdot\}_R$ which are given by the coadjoint orbits of the infinite dimensional Lie group G_R which integrates $\mathfrak{g}_R$. In particular, we shall consider the coadjoint action of G_R on the class $\mathfrak{p}_*$ of Hilbert-Schmidt operators $\boldsymbol{K} \in \mathfrak{g}$ with so-called single-pair kernels [GK]. By definition, a Hilbert-Schmidt operator $\boldsymbol{K} \in \mathfrak{p}_*$ if and only if its kernel is of the form

$$K(x,y) = \begin{cases} a(x)b(y), & x \leq y \\ a(y)b(x), & x > y, \end{cases} \tag{2.27}$$

where a and b are functions on $\mathbb{R}$. (Note that a and b are not necessarily in $L^2(\mathbb{R})$.) In order to describe G_R, we begin by introducing the Lie subgroups (see Remark 2.7)

$$\begin{aligned} &\mathcal{L} = \boldsymbol{I} + \mathfrak{l}, \\ &\mathcal{K} = \{\, k \in G \mid k \circ k^* = k^* \circ k = \boldsymbol{I} \,\} \end{aligned} \tag{2.28}$$

of G which corresponds to the Lie algebras $\mathfrak{l}$ and $\mathfrak{k}$.

Remark 2.7. It is clear that the group operation of G is closed on $\mathcal{L}$. On the other hand, if $\boldsymbol{A} \in \mathfrak{l}$, we can show that the Neumann series $\sum_{j=0}^{\infty}(-1)^j \boldsymbol{A}^j$ converges by using the estimate $\|\boldsymbol{A}^{j+1}\| \leq \|\boldsymbol{A}\|_2^{j+1}/\sqrt{(j-1)!}$, $j = 1, 2, \ldots$, which we can derive from eqn. (8), Section 2.7 of [Sm] provided we interpret the inequality there as being valid almost everywhere under our weaker assumption. Thus $\boldsymbol{I} + \boldsymbol{A}$ is invertible and $(\boldsymbol{I} + \boldsymbol{A})^{-1} = \sum_{j=0}^{\infty}(-1)^j \boldsymbol{A}^j \in \mathcal{L}$. This shows that $\mathcal{L}$ is a subgroup of G. That $\mathcal{L}$ is a Lie subgroup of G now follows since the former is clearly a submanifold of the latter.

Let

$$G_R = \{g \in G \mid g = g_- \circ g_+^{-1}, \text{ where } g_- \in \mathcal{L},\, g_+ \in \mathcal{K}\,\}. \tag{2.29}$$

Then following the procedure in [DLT], we can endow G_R with a Lie group structure by defining the multiplication

$$g * h \equiv g_- \circ h \circ g_+^{-1} \tag{2.30}$$

and we can show that $(G_R, *)$ is a Lie group which corresponds to the Lie algebra $\mathfrak{g}_R$. Moreover, the adjoint action of G_R on $\mathfrak{g}_R$ is given by

$$Ad_{G_R}(g)\boldsymbol{K} = g_- \circ \Pi_{\mathfrak{l}}\boldsymbol{K} \circ g_-^{-1} + g_+ \circ \Pi_{\mathfrak{k}}\boldsymbol{K} \circ g_+^{-1}. \tag{2.31}$$

Hence an easy computation using (2.11) shows that

$$Ad^*_{G_R}(g^{-1})\boldsymbol{K} = \Pi^*_{\mathfrak{l}}(g_- \circ \boldsymbol{K} \circ g_-^{-1}) + \Pi^*_{\mathfrak{k}}(g_+ \circ \boldsymbol{K} \circ g_+^{-1}) \tag{2.32}$$

and the symplectic leaves of $\{\cdot,\cdot\}_R$ are the orbits of this coadjoint action.

Proposition 2.8. *The class $\mathfrak{p}_* \subset \mathfrak{p}$ of Hilbert-Schmidt operators with single-pair kernels is invariant under $Ad^*_{G_R}$.*

Proof. Take $\boldsymbol{K} \in \mathfrak{p}_*$,

$$\boldsymbol{K} \doteq K(x,y) = \begin{cases} a(x)b(y), & x \le y \\ a(y)b(x), & x > y. \end{cases}$$

Then for $g = g_- \circ g_+^{-1} \in G_R$, it is clear that $g_+^{-1} \circ \boldsymbol{K} \circ g_+ \in \mathfrak{p}$. Therefore, $\Pi^*_{\mathfrak{k}}(g_+^{-1} \circ \boldsymbol{K} \circ g_+) = 0$ so that

$$Ad^*_{G_R}(g)\boldsymbol{K} = \Pi^*_{\mathfrak{l}}(g_-^{-1} \circ \boldsymbol{K} \circ g_-).$$

Now, by a straightforward computation using the form of $K(x,y)$ above and the fact that $g_- \in \mathcal{L}$, we can show

$$\Pi^*_{\mathfrak{l}}(g_-^{-1} \circ \boldsymbol{K} \circ g_-) \doteq \begin{cases} (g_-^{-1}a)(x)(g_-^* b)(y), & x \le y \\ (g_-^{-1}a)(y)(g_-^* b)(x), & x > y \end{cases}$$

from which we conclude that $Ad^*_{G_R}(g)\boldsymbol{K} \in \mathfrak{p}_*$, as asserted. □

From this result, it follows that the coadjoint orbit of G_R through an element $\boldsymbol{K} \in \mathfrak{p}_*$ will consist entirely of elements from $\mathfrak{p}_*$. In particular, this means that if the initial data of (2.24) is a single-pair kernel, then $K(x,y;t)$ is also a single-pair kernel for all t. This is the fact which underlies the geometry behind our application in Section 4 below.

Remark 2.9. (a) As the reader will see in Theorem 3.1, every $g \in G$ admits a unique factorization $g = g_- \circ g_+^{-1}$ with $g_- \in \mathcal{L}$ and $g_+ \in \mathcal{K}$. Thus the underlying

manifold of $(G_R, *)$ is just G in this case. Note that this factorization result in Theorem 3.1 is nothing but the global version of the decomposition in Proposition 2.2.

(b) As a final remark of this section, we would like to point out that everything we have done in this section can be pushed through in the more general setting when the Hilbert space is taken to be $L^2(\mathbb{R}, d\mu)$ for an arbitrary Borel measure μ on $\mathbb{R}$.

3. Solution by factorization.

We recall the Lie subgroups

$$\begin{aligned} \mathcal{L} &= \boldsymbol{I} + \mathfrak{l}, \\ \mathcal{K} &= \{\, k \in G \mid k \circ k^* = k^* \circ k = \boldsymbol{I} \,\} \end{aligned} \tag{3.1}$$

of G introduced in Section 2. As the reader will see, they play an important role in the solution of (2.24). In order to discuss the factorization problem, let us also recall several formulas from the theory of regularized determinants which we are going to need in our context. The reader is referred to [S], [Sm] for more details.

Let $\boldsymbol{A}$ be a Hilbert-Schmidt operator on $\mathcal{H} = L^2(\mathbb{R})$, then

$$\mathcal{R}_2(\boldsymbol{A}) := (\boldsymbol{I} + \boldsymbol{A})e^{-\boldsymbol{A}} - \boldsymbol{I} \tag{3.2}$$

is of trace class. Following [S], we can define the regularized determinant

$$\begin{aligned} \det{}_2(\boldsymbol{I} + \boldsymbol{A}) &:= \det(\boldsymbol{I} + \mathcal{R}_2(\boldsymbol{A})) \\ &:= \sum_{k=0}^{\infty} tr \wedge^k (\mathcal{R}_2(\boldsymbol{A})) \end{aligned} \tag{3.3}$$

which obeys the estimate

$$|\det{}_2(\boldsymbol{I} + \boldsymbol{A})| \leq \exp(\|\mathcal{R}_2(\boldsymbol{A})\|_1) \tag{3.4}$$

where $\|\mathcal{R}_2(\boldsymbol{A})\|_1 = tr(\sqrt{\mathcal{R}_2(\boldsymbol{A})^* \mathcal{R}_2(\boldsymbol{A})}\,)$. Suppose $\boldsymbol{I} + \boldsymbol{K} \in G$, then the analog of the first Fredholm minor is given by

$$D_2(\boldsymbol{K}) = -\boldsymbol{K}(\boldsymbol{I} + \boldsymbol{K})^{-1} \det{}_2(\boldsymbol{I} + \boldsymbol{K}) \tag{3.5}$$

and so we have the formula

$$(\boldsymbol{I} + \boldsymbol{K})^{-1} = \boldsymbol{I} + \frac{D_2(\boldsymbol{K})}{\det_2(\boldsymbol{I} + \boldsymbol{K})}. \tag{3.6}$$

In connection with (3.6) above, it is important to note the Plemelj-Smithies formulas

$$\det{}_2(\boldsymbol{I}+\boldsymbol{K}) = 1 + \sum_{m=1}^{\infty} \alpha_m^{(2)}(\boldsymbol{K}) \tag{3.7}$$

and

$$D_2(\boldsymbol{K}) = \boldsymbol{K} + \sum_{m=1}^{\infty} \beta_m^{(2)}(\boldsymbol{K}). \tag{3.8}$$

Here,

$$\alpha_m^{(2)}(\boldsymbol{K}) = \frac{1}{m!} \det \begin{pmatrix} 0 & m-1 & 0 & \cdots & 0 \\ \sigma_2(\boldsymbol{K}) & 0 & m-2 & \cdots & 0 \\ \sigma_3(\boldsymbol{K}) & \sigma_2(\boldsymbol{K}) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_{m-1}(\boldsymbol{K}) & \sigma_{m-2}(\boldsymbol{K}) & \sigma_{m-3}(\boldsymbol{K}) & \cdots & 1 \\ \sigma_m(\boldsymbol{K}) & \sigma_{m-1}(\boldsymbol{K}) & \sigma_{m-2}(\boldsymbol{K}) & \cdots & 0 \end{pmatrix} \tag{3.9}$$

and

$$\beta_m^{(2)}(\boldsymbol{K}) = \frac{1}{m!} \det \begin{pmatrix} \boldsymbol{K} & m & 0 & \cdots & 0 \\ \boldsymbol{K}^2 & 0 & m-1 & \cdots & 0 \\ \boldsymbol{K}^3 & \sigma_2(\boldsymbol{K}) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{K}^m & \sigma_{m-1}(\boldsymbol{K}) & \sigma_{m-2}(\boldsymbol{K}) & \cdots & 1 \\ \boldsymbol{K}^{m+1} & \sigma_m(\boldsymbol{K}) & \sigma_{m-1}(\boldsymbol{K}) & \cdots & 0 \end{pmatrix} \tag{3.10}$$

where

$$\sigma_j(\boldsymbol{K}) = tr(\boldsymbol{K}^j), \quad j \geq 2. \tag{3.11}$$

We next introduce a piece of notation. If $\boldsymbol{K} \in \mathfrak{g}$ and $y \in \mathbb{R}$, we shall denote by $\boldsymbol{K} \mid_{(-\infty,y)}$ the operator $L^2(-\infty,y) \longrightarrow L^2(-\infty,y)$ defined by

$$(\boldsymbol{K} \mid_{(-\infty,y)} \varphi)(x) = \int_{-\infty}^{y} K(x,z)\varphi(z)\,dz \tag{3.12}$$

for $\varphi \in L^2(-\infty,y)$. Similarly, $(\boldsymbol{I}+\boldsymbol{K}) \mid_{(-\infty,y)}$ has an analogous meaning.

In order to solve the integro-differential equation (2.24), the following result is basic.

Theorem 3.1. *Suppose $\boldsymbol{I}+\boldsymbol{K} \in G$, then $\boldsymbol{I}+\boldsymbol{K}$ has a unique factorization*

$$\boldsymbol{I}+\boldsymbol{K} = b_- \circ b_+^{-1} \tag{3.13}$$

where $b_- \in \mathcal{L}$ and $b_+ \in \mathcal{K}$. If $(b_-^{-1})^* - \boldsymbol{I} \doteq C_+(x,y)$, then explicitly, $C_+(x,y)$ is given by

$$\begin{aligned} C_+(x,y) &= -\left(((\boldsymbol{I}+\boldsymbol{S})|_{(-\infty,y)})^{-1} S(\cdot,y)\right)(x), \quad x<y \\ &= -S(x,y) - \frac{(D_2(\boldsymbol{S}|_{(-\infty,y)})S(\cdot,y))(x)}{\det_2((\boldsymbol{I}+\boldsymbol{S})|_{(-\infty,y)})} \end{aligned} \tag{3.14}$$

and $C_+(x,y)=0$ for $y<x$ where $S(x,y)$ is the kernel of

$$\boldsymbol{S} = \boldsymbol{K} + \boldsymbol{K}^* + \boldsymbol{K} \circ \boldsymbol{K}^*. \tag{3.15}$$

Proof. By the analog of Remark 2.7,

$$\mathcal{U} = \boldsymbol{I} + \mathfrak{u}$$

is the Lie group corresponding to the Lie algebra $\mathfrak{u}$ introduced at the end of Section 2. For each $y \in \mathbb{R}$, consider the equation

$$C(x,y) + \int_{-\infty}^{y} S(x,z)C(z,y)\,dz = -S(x,y), \quad x<y. \tag{3.16}$$

Since $\boldsymbol{I}+\boldsymbol{S} = (\boldsymbol{I}+\boldsymbol{K}) \circ (\boldsymbol{I}+\boldsymbol{K})^*$ is positive definite, it follows that $(\boldsymbol{I}+\boldsymbol{S})|_{(-\infty,y)}$ is invertible. Hence (3.16) has a unique solution given by

$$\begin{aligned} C_+(x,y) &= -\left(((\boldsymbol{I}+\boldsymbol{S})|_{(-\infty,y)})^{-1} S(\cdot,y)\right)(x), \quad x<y \\ &= -S(x,y) - \frac{(D_2(\boldsymbol{S}|_{(-\infty,y)})S(\cdot,y))(x)}{\det_2((\boldsymbol{I}+\boldsymbol{S})|_{(-\infty,y)})} \end{aligned}$$

where we have used the formula in (3.6). Set $C_+(x,y)=0$ for $y<x$ and let $\boldsymbol{C}_+$ denote the corresponding operator in $\mathfrak{u}$. Then from (3.16), we have

$$\boldsymbol{C}_+ + \Pi_+(\boldsymbol{S} \circ \boldsymbol{C}_+) = -\boldsymbol{S}_+ \tag{3.17}$$

where $\Pi_+ : \mathfrak{g} \longrightarrow \mathfrak{u}$ is the projection operator to $\mathfrak{u}$ relative to the splitting in (2.26) and $\boldsymbol{S}_+ = \Pi_+\boldsymbol{S}$. But on the other hand, we find

$$\begin{aligned} &(1-\Pi_+)(\boldsymbol{S} + \boldsymbol{S} \circ \boldsymbol{C}_+) - \boldsymbol{S} \\ =&\,(\boldsymbol{I}+\boldsymbol{S}) \circ \boldsymbol{C}_+. \end{aligned} \tag{3.18}$$

Hence it follows that

$$(\boldsymbol{I}+\boldsymbol{S}) \circ (\boldsymbol{I}+\boldsymbol{C}_+) = \boldsymbol{I} + \boldsymbol{B}_-$$

where

$$\boldsymbol{B}_- := \Pi_-(\boldsymbol{S} + \boldsymbol{S} \circ \boldsymbol{C}_+) \in \mathfrak{l}. \tag{3.19}$$

Set $b_- = \boldsymbol{I}+\boldsymbol{B}_-$ and $c_+ = \boldsymbol{I}+\boldsymbol{C}_+$. Then $\boldsymbol{I}+\boldsymbol{S} = b_-\circ c_+^{-1}$. But from $(\boldsymbol{I}+\boldsymbol{S})^* = \boldsymbol{I}+\boldsymbol{S}$, we also have $\boldsymbol{I}+\boldsymbol{S} = (c_+^{-1})^*\circ b_-^*$. Equating the two expression for $\boldsymbol{I}+\boldsymbol{S}$, we find

$$c_+^*\circ b_- = b_-^*\circ c_+ \in \mathcal{L}\cap\mathcal{U}.$$

As $\mathcal{L}\cap\mathcal{U} = \boldsymbol{I}$, we conclude that $c_+ = (b_-^*)^{-1}$ and so we have established the factorization

$$\boldsymbol{I}+\boldsymbol{S} = b_-\circ b_-^*.$$

Now we define

$$b_+ = (\boldsymbol{I}+\boldsymbol{K})^{-1}\circ b_-.$$

Then a straightforward verification shows that $b_+\in\mathcal{K}$. Finally, the uniqueness of the factors $b_\pm$ is obvious. □

Theorem 3.2. *Let $\boldsymbol{K}_0\in\mathfrak{g}$ and let $b_-(t)\in\mathcal{L}$, $b_+(t)\in\mathcal{K}$ be the unique solution of the factorization problem*

$$\exp\left(-\frac{1}{2}t\boldsymbol{K}_0\right) = b_-(t)\circ b_+(t)^{-1}. \tag{3.20}$$

Then for all t,

$$\boldsymbol{K}(t) = b_\pm(t)^{-1}\circ\boldsymbol{K}_0\circ b_\pm(t) \tag{3.21}$$

solves the initial value problem

$$\dot{\boldsymbol{K}} = \frac{1}{2}[\,\Pi_{\mathfrak{l}}\,\boldsymbol{K},\boldsymbol{K}\,] = \frac{1}{2}[\,\boldsymbol{K},\Pi_{\mathfrak{k}}\,\boldsymbol{K}],\quad \boldsymbol{K}(0)=\boldsymbol{K}_0. \tag{3.22}$$

Proof. We shall present a direct proof of the theorem. First of all, the factorization problem in (3.20) has unique solutions $b_-(t)\in\mathcal{L}$ and $b_+(t)\in\mathcal{K}$ by Theorem 3.1. Take

$$\boldsymbol{K}(t) = b_+(t)^{-1}\circ\boldsymbol{K}_0\circ b_+(t).$$

By differentiating the expression, we have

$$\dot{\boldsymbol{K}}(t) = [\,\boldsymbol{K}(t),\, b_+(t)^{-1}\circ\dot{b}_+(t)\,]. \tag{$*$}$$

On the other hand, by differentiating (3.20), we find

$$-\frac{1}{2}\boldsymbol{K}(t) = b_-(t)^{-1}\circ\dot{b}_-(t) - b_+(t)^{-1}\circ\dot{b}_+(t).$$

Hence by applying $\Pi_{\mathfrak{k}}$ to both sides of the above expression, the result is

$$b_+(t)^{-1} \circ \dot{b}_+(t) = \frac{1}{2}\Pi_{\mathfrak{k}}\, \boldsymbol{K}(t).$$

Therefore, on substituting into (*), we conclude that

$$\dot{\boldsymbol{K}}(t) = \frac{1}{2}[\,\boldsymbol{K}(t), \Pi_{\mathfrak{k}}\, \boldsymbol{K}(t)\,].$$

This shows that $\boldsymbol{K}(t) = b_+(t)^{-1} \circ \boldsymbol{K}_0 \circ b_+(t)$ solves the initial value problem. $\square$

Remark 3.3. (a) The factorization method is the most important feature of classical r-matrix theory. It should be emphasized that there is no universal method to solve the factorization problems. Rather, the method varies with the Lie groups involved. For examples involving finite dimensional matrix groups, the reader is referred to [DLT] for further information. On the other hand, factorization problems associated with loop groups are related to Riemann-Hilbert problems. See, for example, [RSTS] and [DL] in this connection.
(b) We can also give a geometric proof of Theorem 3.2 by using Poisson reduction. Indeed, from this point of view, we can understand the Hamiltonian flow in (3.21) as the projection of some simple Hamiltonian flow on the cotangent bundle T^*G. We will give a sketch of the argument here, following essentially the outline on p. 180 of [STS]. Let $(G_R, *)$ be the Lie group introduced in Section 2 with Lie algebra $\mathfrak{g}_R$. Consider the action of G_R on G,

$$G_R \times G \longrightarrow G:\ g \cdot h = g_- \circ h \circ g_+^{-1},\ g \in G_R, h \in G. \tag{3.23}$$

We can lift this up to obtain a canonical action on T^*G [AM]. Indeed, this lifted action is given by

$$G_R \times T^*G \longrightarrow T^*G : g \cdot (h, \boldsymbol{K}) = (g_- \circ h \circ g_+^{-1}, Ad_G^*(g_+^{-1})\boldsymbol{K}) \tag{3.24}$$

where we have made the identification $T^*G \simeq G \times \mathfrak{g}^* \simeq G \times \mathfrak{g}$ by using left translation and the pairing on $\mathfrak{g}$. Consequently, by Poisson reduction, the orbit space $T^*G/G_R \simeq \mathfrak{g}^* \simeq \mathfrak{g}$ has a unique Poisson structure such that the canonical projection

$$\pi : T^*G \longrightarrow T^*G/G_R \simeq \mathfrak{g}, (g, \boldsymbol{K}) \mapsto Ad_G^*(g_+)\boldsymbol{K} = g_+^{-1} \circ \boldsymbol{K} \circ g_+ \tag{3.25}$$

is a Poisson map. By a direct computation, we can show that the Poisson structure on $\mathfrak{g} \simeq T^*G/G_R$ is nothing but $\{\cdot,\cdot\}_R$. Now we consider the bi-invariant Hamiltonian on $T^*G \simeq G \times \mathfrak{g}$, given by

$$\widehat{H}_1(g, \boldsymbol{K}) = H_1(\boldsymbol{K}). \tag{3.26}$$

Then its Hamiltonian equations of motion are

$$\begin{aligned} \dot{g} &= -\frac{1}{2} g \circ \boldsymbol{K}, \\ \dot{\boldsymbol{K}} &= 0. \end{aligned} \tag{3.27}$$

Therefore, if we denote the corresponding flow by F_t, we have in particular that

$$F_t(\boldsymbol{I}, \boldsymbol{K}_0) = \left(\exp\left(-\frac{1}{2} t \boldsymbol{K}_0\right), \boldsymbol{K}_0\right). \tag{3.28}$$

Consequently, the Hamiltonian flow generated by the reduced Hamiltonian $H_{red.} = H_1$ on the orbit space $T^*G/G_R \simeq \mathfrak{g}$ is given by

$$\begin{aligned} \bar{F}_t(\boldsymbol{K}_0) &= \pi \circ F_t(\boldsymbol{I}, \boldsymbol{K}_0) \\ &= Ad^*_G(b_+(t))\boldsymbol{K}_0 \\ &= b_+(t)^{-1} \circ \boldsymbol{K}_0 \circ b_+(t), \end{aligned} \tag{3.29}$$

as required.

We are now going to write down the solution of the integro-differential equation (2.24) by combining the above theorems. For this purpose, let $\boldsymbol{K}_0 \doteq K(x,y)$, $\boldsymbol{K}(t) \doteq K(x,y;t)$ and write

$$\exp(-t\boldsymbol{K}_0) = \boldsymbol{I} + \boldsymbol{S}(t), \quad \boldsymbol{S}(t) \doteq S(x,y;t). \tag{3.30}$$

By Theorem 3.1, the solution $b_-(t)$ of the factorization problem (3.20) is such that

$$(b_-(t)^{-1})^* - \boldsymbol{I} = \boldsymbol{C}_+^t \doteq C_+(x,y;t) \tag{3.31}$$

where

$$\begin{aligned} C_+(x,y;t) &= -\left((e^{-t\boldsymbol{K}_0}|_{(-\infty,y)})^{-1} S(\cdot,y;t)\right)(x)\, \chi_{(x,\infty)}(y) \\ &= -\left(S(x,y;t) + \frac{(D_2(\boldsymbol{S}(t)|_{(-\infty,y)})S(\cdot,y;t))(x)}{\det_2\left(e^{-t\boldsymbol{K}_0}|_{(-\infty,y)}\right)}\right)\chi_{(x,\infty)}(y). \end{aligned} \tag{3.32}$$

On the other hand, it follows from the proof of Theorem 3.1 (see (3.19) and the definition of b_-) that

$$b_-(t) = \boldsymbol{I} + \Pi_-(\boldsymbol{S}(t) + \boldsymbol{S}(t) \circ \boldsymbol{C}_+^t) \tag{3.33}$$

and hence

$$b_-(t) - \boldsymbol{I} \doteq \left(S(x,y;t) + \int_{-\infty}^{\infty} S(x,z;t) C_+(z,y;t) dz \right) \chi_{(-\infty,x)}(y). \tag{3.34}$$

Consequently, from

$$\begin{aligned} \boldsymbol{K}(t) &= b_-(t)^{-1} \circ \boldsymbol{K}_0 \circ b_-(t) \\ &= \boldsymbol{K}_0 + (b_-(t)^{-1} - \boldsymbol{I}) \circ \boldsymbol{K}_0 + \big(\boldsymbol{K}_0 + (b_-(t)^{-1} - \boldsymbol{I}) \circ \boldsymbol{K}_0\big) \circ (b_-(t) - \boldsymbol{I}) \end{aligned} \tag{3.35}$$

and (3.31), (3.34), we find

$$\begin{aligned} K(\xi,\eta;t) = K(\xi,\eta) &+ \int_{-\infty}^{\infty} C_+(\zeta,\xi;t) K(\zeta,\eta)\, d\zeta \\ &+ \int_{\eta}^{\infty} \left(K(\xi,\zeta_2) + \int_{-\infty}^{\infty} C_+(\zeta_1,\xi;t) K(\zeta_1,\zeta_2)\, d\zeta_1 \right) \\ &\quad \cdot \left(S(\zeta_2,\eta;t) + \int_{-\infty}^{\infty} S(\zeta_2,\zeta_3;t) C_+(\zeta_3,\eta;t)\, d\zeta_3 \right) d\zeta_2. \end{aligned} \tag{3.36}$$

Remark 3.4. In a similar fashion, we can write down the solution of the integro-differential equation (2.23). Indeed, all we have to do is to replace $e^{-t\boldsymbol{K}_0}$ in (3.30) and (3.32) by $\exp\left(-\frac{1}{2}t\boldsymbol{K}_0\right) \circ \exp\left(-\frac{1}{2}t(\boldsymbol{K}_0)^*\right)$. The operator $\boldsymbol{S}(t)$ and its kernel $S(x,y;t)$ are of course much more complicated in this case.

4. Solution of the Camassa-Holm equation.

In this section, we shall consider the Camassa-Holm (CH) equation in the non-dispersive case:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + u u_{xxx}. \tag{4.1}$$

We will begin with a sketch of the connection between (4.1) and the integro-differential equation (2.24), as first discovered by Camassa [C1]. To do so, we introduce the auxiliary variable

$$m(x,t) = (1 - \partial_x^2) u(x,t) \tag{4.2}$$

and rewrite the CH equation in the form

$$m_t + u m_x = -2 m u_x. \tag{4.3}$$

We shall make the following assumption on the initial data:
(i) $u_0 = u(\cdot, 0)$ is in the Schwarz class $\mathcal{S}(\mathbb{R})$,
(ii) $m(x,0) = u_0(x) - u_0''(x) > 0$ for all $x \in \mathbb{R}$.

In the Lagrangian point of view of the CH equation, we consider the trajectory $q(\xi, t)$ of a fluid particle which at $t = 0$ is located at $\xi \in \mathbb{R}$. Thus we have

$$\dot{q}(\xi, t) = u(q(\xi, t), t), \quad q(\xi, 0) = \xi \tag{4.4}$$

and a straightforward calculation (see [C1], [Con]) using (4.3) and (4.4) shows that

$$m(q(\xi, t), t)(q_\xi(\xi, t))^2 = m(\xi, 0). \tag{4.5}$$

Now it follows from [C1],[C2] and [Con] that for the class of initial data introduced above, we have

$$0 < q_\xi(\xi, t) < \infty \tag{4.6}$$

for all $t > 0$. In particular, this means that the trajectories of the fluid particles never cross.

Therefore, if we define

$$w(x, t) = \int_{-\infty}^{x} \sqrt{m(y, t)}\, dy, \tag{4.7}$$

then

$$w(q(\xi, t), t) = w(\xi, 0) = w_0(\xi) \tag{4.8}$$

so that we can rewrite (4.5) as

$$m(q(\xi, t), t) = \left(\frac{w_0'(\xi)}{q_\xi(\xi, t)} \right)^2. \tag{4.9}$$

By introducing the auxiliary function

$$p(\xi, t) = m(q(\xi, t), t) q_\xi(\xi, t) = \frac{(w_0'(\xi))^2}{q_\xi(\xi, t)} \tag{4.10}$$

and using the formula

$$u(x, t) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x - q(\eta, t)|} m(q(\eta, t), t) q_\eta(\eta, t)\, d\eta, \tag{4.11}$$

it follows from (4.4) and (4.9) that

$$\begin{aligned}\dot{q}(\xi,t) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|q(\xi,t)-q(\eta,t)|}p(\eta,t)\,d\eta \\ &= \frac{\delta H}{\delta p},\end{aligned} \tag{4.12}$$

and hence

$$\begin{aligned}\dot{p}(\xi,t) &= \frac{1}{2}p(\xi,t)\int_{\mathbb{R}} sgn(\xi-\eta)e^{-|q(\xi,t)-q(\eta,t)|}p(\eta,t)\,d\eta \\ &= -\frac{\delta H}{\delta q}\end{aligned} \tag{4.13}$$

where

$$H = \frac{1}{4}\int_{\mathbb{R}^2} e^{-|q(\xi,t)-q(\eta,t)|}p(\xi,t)p(\eta,t)\,d\xi d\eta. \tag{4.14}$$

Therefore, (4.12), (4.13) is a Hamiltonian system with constraint given by (4.10).(See Remark 4.1 (b) and the appendix for more details.) If $q(\xi,t)$, $p(\xi,t)$ satisfy (4.12), (4.13) and we define

$$K(\xi,\eta;t) = \frac{1}{2}e^{-\frac{1}{2}|q(\xi,t)-q(\eta,t)|}\sqrt{p(\xi,t)p(\eta,t)}, \tag{4.15}$$

then a direct calculation shows that $K(\xi,\eta;t)$ evolves under the integro-differential equation (2.24). Note that the kernel $K(\cdot,\cdot;t)$ defined in (4.15) is a positive single-pair kernel. Moreover, $K(\cdot,\cdot;t)\in C(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ and the corresponding operator is of trace class.(See Remark 4.1 (b).)

Remark 4.1. (a) For the class of initial data which we consider here, it has been established in [R] that the solution u of the Cauchy problem associated to the CH equation (4.1) satisfies $u\in C^\infty((0,\infty),\mathcal{S}(\mathbb{R}))$.

(b) On the other hand, by the ODE for $q(\xi,t)$ in (4.4) (and Remark 4.1 (a)) or otherwise, we know that $q(\cdot,t)\in C^\infty(\mathbb{R})$ and hence $p(\cdot,t)$ in also in $C^\infty(\mathbb{R})$ by (4.10). Now it follows from (4.12) that

$$\dot{q}_\xi(\xi,t) = \left(-\frac{1}{2}\int_{\mathbb{R}} sgn(\xi-\eta)e^{-|q(\xi,t)-q(\eta,t)|}p(\eta,t)\,d\eta\right)q_\xi(\xi,t).$$

Solving, we find

$$q_\xi(\xi,t) = \exp\left(-\frac{1}{2}\int_0^t\int_{\mathbb{R}} sgn(\xi-\eta)e^{-|q(\xi,\tau)-q(\eta,\tau)|}p(\eta,\tau)\,d\eta\,d\tau\right). \tag{$**$}$$

Since $P = \int_{\mathbb{R}} p(\eta,t)\,d\eta$ is a conserved quantity [C1], we obtain from (**) that $e^{-Pt/2} \le q_\xi(\xi,t) \le e^{Pt/2}$, $\xi \in \mathbb{R}$, $t > 0$. Consequently, for each $t > 0$, the function $q(\cdot,t)$ is a diffeomorphism of the line which is strictly increasing and satisfies $\lim_{\xi\to\pm\infty} q(\xi,t) = \pm\infty$. Moreover, $q(\cdot,t)$ is a tempered distribution. Now, if we continue by differentiating (**) repeatedly, we can show by induction that there exist constants $C_k(t) > 0$ for $t > 0$ such that $|\partial_\xi^k q(\xi,t)| \le C_k(t)$, $k = 2,3,\cdots$. Since $m(\xi,0) = (w'(\xi))^2$ is a rapidly decreasing function, it is now easy to see by using these bounds that $p(\cdot,t)$ as defined in (4.10) is also a rapidly decreasing function.

(c) We will show that the map $(q,p) \mapsto \gamma(q,p) \doteq 2e^{-\frac{1}{2}|q(\xi)-q(\eta)|}\sqrt{p(\xi)p(\eta)}$ is a Poisson map in the appendix. (This is where the discussion in (b) above will be used in setting up the domain of the map.) However, it is important to point out that the image of this map is not invariant under $Ad^*_{G_R}$, although it is invariant under the CH-flow.

To solve the Cauchy problem

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad u(x,0) = u_0(x) \tag{4.16}$$

in the space $\mathcal{S}(\mathbb{R})$ under the above assumptions, we proceed as follows. From the initial data u_0 which satisfies assumptions (i) and (ii) above, we obtain the initial conditions for $q(\xi,t)$ and $p(\xi,t)$:

$$q(\xi,0) = \xi, \quad p(\xi,0) = m(\xi,0) = (w_0'(\xi))^2. \tag{4.17}$$

From this, we obtain the kernel $K(\xi,\eta)$ of the operator $\boldsymbol{K}_0$, namely,

$$K(\xi,\eta) = \frac{1}{2}e^{-\frac{1}{2}|\xi-\eta|}w_0'(\xi)w_0'(\eta). \tag{4.18}$$

Remark 4.2. Note that from our assumption on the initial data u_0 and (4.18) above, it is clear that $\boldsymbol{K}_0 \in \mathfrak{p}_*$. Hence it follows from Proposition 2.8 that the solution $\boldsymbol{K}(t)$ of (3.22) is on the coadjoint orbit $\mathcal{O}_{\boldsymbol{K}_0} = \{\, Ad^*_{G_R}(g)\boldsymbol{K}_0 \mid g \in G\,\} \subset \mathfrak{p}_*$ for all t.

Now we solve the initial value problem (3.22) whose solution $\boldsymbol{K}(t)$ is given by the formula in (3.21) and we denote the kernel corresponding to $\boldsymbol{K}(t)$ by $K(\xi,\eta,;t)$. Since $K(\xi,\eta,;t)$ is given by the formula in (4.15) where $q(\xi,t)$, $p(\xi,t)$ are solutions of (4.12), (4.13) with initial condition given in (4.17), we can recover $p(\eta,t)$ from the formula

$$p(\eta,t) = 2K(\eta,\eta;t). \tag{4.19}$$

From (4.15) and (4.19), it follows that

$$e^{-|q(\xi,t)-q(\eta,t)|} = \frac{K(\xi,\eta;t)^2}{K(\xi,\xi;t)K(\eta,\eta;t)}. \tag{4.20}$$

Hence we obtain

$$\dot{q}(\eta,t) = \int_{\mathbb{R}} \frac{K(\eta,\zeta;t)^2}{K(\eta,\eta;t)}\, d\zeta \tag{4.21}$$

and consequently, we can determine $q(\eta,t)$ from the formula

$$q(\eta,t) = \eta + \int_0^t \left(\int_{\mathbb{R}} \frac{K(\eta,\zeta;\tau)^2}{K(\eta,\eta;\tau)}\, d\zeta \right) d\tau. \tag{4.22}$$

Finally, we obtain the solution of the Cauchy problem (4.16) from

$$u(x,t) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-q(\eta,t)|} p(\eta,t)\, d\eta \tag{4.23}$$

where $p(\eta,t)$ and $q(\eta,t)$ are given in (4.19) and (4.22) above.

Note that the explicit formula for $K(\xi,\eta;t)$ is given in (3.36) where in the present case we can interpret equality in the pointwise sense as the kernels of all operators involved in (3.35) are continuous. Alternatively, we can make use of Mercer's theorem in writing down the explicit formula for $K(\xi,\eta;t)$. In this connection, note that $\boldsymbol{K}_0$ is compact and hence its spectrum $\sigma(\boldsymbol{K}_0)$ is discrete with no limit points except possibly at 0. Moreover, $\boldsymbol{K}_0$ has no negative eigenvalues. To see this, let λ be an eigenvalue of $\boldsymbol{K}_0$ and ϕ a corresponding normalized eigenfunction. Then from

$$\frac{1}{2} \int_{\mathbb{R}} e^{-\frac{1}{2}|\xi-\eta|} w_0'(\xi) w_0'(\eta) \phi(\eta)\, d\eta = \lambda \phi(\xi), \tag{4.24}$$

we have

$$\lambda = \frac{1}{2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-\frac{1}{2}|\xi-\eta|} (w_0'\phi)(\eta)\, d\eta \right) (w_0'\phi)(\xi)\, d\xi. \tag{4.25}$$

Therefore, on applying the Parseval formula and the convolution theorem in Fourier transforms to the right hand of (4.25), we obtain[1]

$$\lambda = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{|\widehat{(w_0'\phi)}(k)|^2}{k^2 + \frac{1}{4}}\, dk \tag{4.26}$$

from which the positivity is clear. (Here $\widehat{w_0'\phi}$ is the Fourier transform of $w_0'\phi$.) Hence this shows that $\boldsymbol{K}_0$ has no negative eigenvalues. To give an estimate of

[1] We owe this argument and the sharpening of (4.28) to the referee.

$\sigma(\boldsymbol{K}_0)$, note that the integral on the right hand side of (4.25) obeys the inequalities

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\frac{1}{2}|\xi-\eta|} w_0'(\eta)\phi(\eta)\, d\eta\right) w_0'(\xi)\phi(\xi)\, d\xi\right| \\
&\leq \left(\int_{\mathbb{R}} w_0'(\xi)|\phi(\xi)|\, d\xi\right)^2 \\
&\leq \int_{\mathbb{R}} (w_0'(\xi))^2\, d\xi = P,
\end{aligned} \tag{4.27}$$

where we have used the Cauchy-Schwarz inequality and the normalization of ϕ in going from the second line to the last one in (4.27) above. Combining the above analysis, we can now conclude that

$$\sigma(\boldsymbol{K}_0) \subset [0, P/2]. \tag{4.28}$$

Let $\sigma(\boldsymbol{K}_0) = \{\lambda_i\}_{i=1}^{\infty}$ and let $\{\phi_i\}_{i=1}^{\infty}$ be the corresponding normalized eigenfunctions. Then we have

$$\boldsymbol{K}_0 = \sum_{i=1}^{\infty} \lambda_i \phi_i \otimes \phi_i \tag{4.29}$$

so that from (3.21), (3.20) and the orthogonality of $b_+(t)$, we find

$$\begin{aligned}
\boldsymbol{K}(t) &= \sum_{i=1}^{\infty} \lambda_i (b_+(t)^{-1}\phi_i) \otimes (b_+(t)^{-1}\phi_i) \\
&= \sum_{i=1}^{\infty} \lambda_i e^{-t\lambda_i} (b_-(t)^{-1}\phi_i) \otimes (b_-(t)^{-1}\phi_i).
\end{aligned} \tag{4.30}$$

Since $\sigma(\boldsymbol{K}(t)) = \sigma(\boldsymbol{K}_0)$, the series $\sum_{i=1}^{\infty} \lambda_i e^{-t\lambda_i}(b_-(t)^{-1}\phi_i)(\xi)(b_-(t)^{-1}\phi_i)(\eta)$ converges absolutely and uniformly on compact sets and it follows from (4.30) and Mercer's theorem that

$$K(\xi, \eta; t) = \sum_{i=1}^{\infty} \lambda_i e^{-t\lambda_i}(b_-(t)^{-1}\phi_i)(\xi)(b_-(t)^{-1}\phi_i)(\eta) \tag{4.31}$$

where from (3.31) and (3.32),

$$(b_-(t)^{-1}\phi_i)(x) = \phi_i(x) - \int_{-\infty}^{x} \left(S(y, x; t) + \frac{(D_2(\boldsymbol{S}(t)|_{(-\infty,x)})S(\cdot, x; t))(y)}{\det_2\left(e^{-t\boldsymbol{K}_0}|_{(-\infty,x)}\right)} \right) \phi_i(y)\, dy. \tag{4.32}$$

Remark 4.3. Under different assumption on $m(\xi, 0)$ and using an entirely different method, the Lagrangian form of the CH equation was integrated in terms of certain

Fredholm determinants in [M]. In particular, the spectral problem in [M] is given by

$$(\frac{1}{4} - \partial_x^2) f(x,t) = \lambda m(x,t) f(x,t). \tag{4.33}$$

On the other hand, our spectral problem reads:

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}|q(\xi,t)-q(\eta,t)|}\sqrt{p(\xi,t)p(\eta,t)}\phi(\eta,t)\,d\eta = \lambda\phi(\xi,t). \tag{4.34}$$

It is a natural question to ask if (4.34) can be derived from (4.33). To put it differently, can we derive the kernel (4.15) (as discovered in [C1]) from (4.33)? As the referee pointed out to us, the answer is an affirmative yes. Indeed, under the assumption in [M], (4.33) can be rewritten as

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}|x-y|} m(y,t) f(y,t)\,dy = \frac{f(x,t)}{\lambda}. \tag{4.35}$$

Then the change to the Lagrangian variable $x = q(\xi,t)$ gives

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}|q(\xi,t)-q(\eta,t)|} p(\eta,t) f(q(\eta,t),t)\,d\eta = \frac{f(q(\xi,t),t)}{\lambda} \tag{4.36}$$

where we have invoked the definition of $p(\cdot,t)$ in (4.10). For our class of initial data, $p(\xi,t) > 0$ for all t. Therefore,if we multiply both sides of (4.36) by $\sqrt{p(\xi,t)}$, and setting $\phi(\xi,t) = f(q(\xi,t),t)\sqrt{p(\xi,t)}$, the result is (4.34) above (modulo a factor $\frac{1}{2}$) provided we replace λ by $\frac{1}{\lambda}$.

Appendix

Let $U = \mathcal{S}'(\mathbb{R}) \cap \{\text{strictly increasing diffeomorphisms } q : \mathbb{R} \longrightarrow \mathbb{R}\}$, where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions, and let V be the subset of $\mathcal{S}(\mathbb{R})$ consisting of those $p \in \mathcal{S}(\mathbb{R})$ which are strictly positive. We equip $U \times V$ with the Poisson bracket

$$\{\mathcal{F}_1, \mathcal{F}_2\}(q,p) = \int_{\mathbb{R}} \left(\frac{\partial \mathcal{F}_1}{\partial q(\xi)} \frac{\partial \mathcal{F}_2}{\partial p(\xi)} - \frac{\partial \mathcal{F}_1}{\partial p(\xi)} \frac{\partial \mathcal{F}_2}{\partial q(\xi)} \right) d\xi. \qquad (A1)$$

For $(q,p) \in U \times V$, put

$$\ell_+(\xi) = \sqrt{2p(\xi)}\, e^{\frac{1}{2}q(\xi)}, \;\; \ell_-(\xi) = \sqrt{2p(\xi)}\, e^{-\frac{1}{2}q(\xi)}. \qquad (A2)$$

Proposition A. *The map $\gamma : U \times V \longrightarrow \mathfrak{g}_R^* \simeq \mathfrak{g}_R$ defined by*

$$\gamma(q,p) \doteq 2\, e^{-\frac{1}{2}|q(\xi)-q(\eta)|} \sqrt{p(\xi)p(\eta)} = \begin{cases} \ell_+(\xi)\ell_-(\eta), & \xi \le \eta \\ \ell_+(\eta)\ell_-(\xi), & \xi > \eta, \end{cases} \qquad (A3)$$

is a Poisson map.

Proof. We want to show

$$\{F_1 \circ \gamma, F_2 \circ \gamma\}(q,p) = \{F_1, F_2\}_R(\gamma(q,p)).$$

To compute the Fréchet derivatives, we proceed as follows. First of all,

$$\int_{\mathbb{R}} \frac{\partial (F_i \circ \gamma)}{\partial q(\xi)} \widetilde{q}(\xi)\, d\xi = \left(dF_i(\gamma(q,p)), \frac{d}{d\epsilon}\Big|_{\epsilon=0} \gamma(q + \epsilon \widetilde{q}, p) \right). \qquad (\dagger)$$

By using the definition of γ, a straight forward computation shows that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \gamma(q + \epsilon\widetilde{q}, p) \doteq \begin{cases} \frac{1}{2}\ell_+(\xi)\ell_-(\eta)(\widetilde{q}(\xi) - \widetilde{q}(\eta)), & \xi \le \eta, \\ \frac{1}{2}\ell_+(\eta)\ell_-(\xi)(\widetilde{q}(\eta) - \widetilde{q}(\xi)), & \xi > \eta. \end{cases}$$

Substitute this into ($\dagger$), it follows after some manipulation that

$$\frac{\partial (F_i \circ \gamma)}{\partial q(\xi)} = \frac{1}{2}\ell_+(\xi)((\Pi_{\mathfrak{l}}\, dF_i(\gamma))^* \ell_-)(\xi) - \ell_-(\xi)((\Pi_{\mathfrak{l}}\, dF_i(\gamma))\ell_+)(\xi)$$

where $dF_i(\gamma)$ is the shorthand for $dF_i(\gamma(q,p))$ which we will use from now onwards. In a similar way, we can show that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \gamma(q, p + \epsilon\widetilde{p}) \doteq \begin{cases} \frac{\ell_-(\eta)}{\ell_-(\xi)}\widetilde{p}(\xi) + \frac{\ell_+(\xi)}{\ell_+(\eta)}\widetilde{p}(\eta), & \xi \le \eta, \\ \frac{\ell_-(\xi)}{\ell_-(\eta)}\widetilde{p}(\eta) + \frac{\ell_+(\eta)}{\ell_+(\xi)}\widetilde{p}(\xi), & \xi > \eta, \end{cases}$$

and
$$\frac{\partial(F_i \circ \gamma)}{\partial p(\xi)} = \frac{1}{\ell_+(\xi)}((\Pi_{\mathfrak{l}}\, dF_i(\gamma))\ell_+)(\xi) + \frac{1}{\ell_-(\xi)}((\Pi_{\mathfrak{l}}\, dF_i(\gamma))^*\ell_-)(\xi).$$

Next, we substitute the Fréchet derivatives into the expression for the Poisson bracket between $F_1 \circ \gamma$ and $F_2 \circ \gamma$, this gives

$$\begin{aligned}
&\{F_1 \circ \gamma, F_2 \circ \gamma\}(q,p)\\
=&\frac{1}{2}\int_{\mathbb{R}} [\,((\Pi_{\mathfrak{l}}\, dF_1(\gamma))^*\ell_-)(\xi)((\Pi_{\mathfrak{l}}\, dF_2(\gamma))\ell_+)(\xi)\\
&\quad - ((\Pi_{\mathfrak{l}}\, dF_1(\gamma))\ell_+)(\xi)((\Pi_{\mathfrak{l}}\, dF_2(\gamma))^*\ell_-)(\xi)\\
&\quad - (F_1 \leftrightarrow F_2)\,]\, d\xi\\
=&\int_{\mathbb{R}} [\,((\Pi_{\mathfrak{l}}\, dF_1(\gamma))^*\ell_-)(\xi)((\Pi_{\mathfrak{l}}\, dF_2(\gamma))\ell_+)(\xi)\\
&\quad - ((\Pi_{\mathfrak{l}}\, dF_1(\gamma))\ell_+)(\xi)((\Pi_{\mathfrak{l}}\, dF_2(\gamma))^*\ell_-)(\xi)\,]\, d\xi.
\end{aligned}$$

But now by using the fact that $\Pi_{\mathfrak{l}}\, dF_1(\gamma)$, $\Pi_{\mathfrak{l}}\, dF_2(\gamma) \in \mathfrak{l}$, we can show that

$$\int_{\mathbb{R}} ((\Pi_{\mathfrak{l}}\, dF_1(\gamma))^*\ell_-)(\xi)((\Pi_{\mathfrak{l}}\, dF_2(\gamma))\ell_+)(\xi)\, d\xi = (\gamma(q,p), \Pi_{\mathfrak{l}}\, dF_1(\gamma) \circ \Pi_{\mathfrak{l}}\, dF_2(\gamma)).$$

Interchanging the indices 1 and 2, we obtain

$$\int_{\mathbb{R}} ((\Pi_{\mathfrak{l}}\, dF_2(\gamma)^*\ell_-)(\xi)((\Pi_{\mathfrak{l}}\, dF_1(\gamma)\ell_+)(\xi)\, d\xi = (\gamma(q,p), \Pi_{\mathfrak{l}}\, dF_2(\gamma) \circ \Pi_{\mathfrak{l}}\, dF_1(\gamma)).$$

Therefore, when we subtract the second expression from the first, the result is

$$\{F_1 \circ \gamma, F_2 \circ \gamma\}(q,p) = (\gamma(q,p), [\Pi_{\mathfrak{l}}\, dF_1(\gamma), \Pi_{\mathfrak{l}}\, dF_2(\gamma)]). \qquad (\ddagger)$$

To complete the proof, note that $(\gamma(q,p), [\Pi_{\mathfrak{k}}\, dF_1(\gamma), \Pi_{\mathfrak{k}}\, dF_2(\gamma)]) = 0$ as $\gamma(q,p) \in \mathfrak{p}$. Consequently, when we add $-(\gamma(q,p), [\Pi_{\mathfrak{k}}\, dF_1(\gamma), \Pi_{\mathfrak{k}}\, dF_2(\gamma)])$ to the right hand side of ($\ddagger$), the assertion follows. $\square$

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